LETTER TO THE EDITOR

The billiard algorithm and KS entropy

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Abstract. There is known to be a close relation between the Kolmogorov-Sinai entropy (sum of the positive Lyapunov exponents) of an ergodic dynamical system and the algorithmic complexity of encoding trajectories of the system with respect to some partition. In this letter, we explicitly give an encoding which demonstrates this relation for the square Sinai billiard. The encoding depends on the fact that the collision criterion for the billiard is an example of rational approximants. The method may be used to achieve very fast simulation times for the system.

There exists a theoretical relation [1] between the most efficient encoding of a trajectory of a dynamical system with respect to a partition and the Kolmogorov-Sinai (KS) entropy of the same system. Unfortunately, there is in general no prescription for devising this efficient encoding. Here we explicitly give a prescription for encoding trajectories for the square Sinai billiard, thus achieving the theoretical relation (to within an additive constant). This result may be used to attain very fast simulation times for the billiard system as we demonstrate numerically.

In a seminal work, Brudno [1] equated the measure theoretic entropy of a dynamical system and the algorithmic complexity of describing the symbol sequence (with respect to a generating partition) of almost all trajectories. More precisely, he showed

$$K(x, T) = h_\mu(T) \tag{1}$$

holds for μ-almost all $x \in X$. Here T is a map with invariant measure μ, and x is a point of a topological space X. The function K is defined as the lowest bound of the ratio of the shortest program needed to compute l bits of the symbol sequence with respect to some partition, divided by l (as $l \to \infty$). On the other hand, the entropy h of a map T is given by

$$h = \int \ln|J_+| \, d\mu \tag{2}$$

where J_+ represents the Jacobian of the expanding subspace of the map [2]†. Intuitively speaking, the Brudno result states that the average loss of information in the system per iteration of the map T is equal to the average information per bit of the most compact coding of the trajectory. We proceed to give an encoding for the billiard system, that demonstrates equation (1).

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† This expression for h is valid for very general conditions on the map T and the space X.

We consider the one-parameter family of dynamical systems, $\text{SSB}(R)$, where the parameter R characterizes the square Sinai billiard for each radius R, where the lattice constant is normalized to unity. Figure 1 depicts the trajectory of a point particle in the universal covering space. The point particle makes elastic collisions with the circular scatterers and travels linearly between collisions. Let us take a close look at the collision with a certain scatterer which we take to be centred at the origin. Each collision may be characterized by two angles, the α and η of figure 2. In order to find the next scatterer, one searches for the 'closest' integers M, N that satisfy

$$R \geqslant |S_{M,N}| \qquad S_{M,N} = R \sin \alpha + M \sin \eta - N \cos \eta \qquad (3)$$

where R is the radius of the scatterer. The quantity $|S_{M,N}|$ is the shortest distance that the point particle trajectory makes with a lattice site given by (M, N). By 'closest' we mean that of all the pairs of integers satisfying (3), the quantity $M^2 + N^2$ is a minimum.

Our first assertion is that M and N for a collision event must be relatively prime. This is easy to prove from (3) and follows from straightforward geometric consider-ations. Our second observation is that the integer pair M, N (that is, N/M) forms a
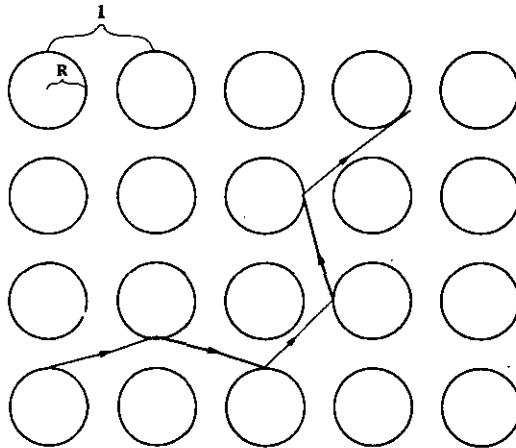

Figure 1. The square Sinai billiard. A point particle moving with velocity and energy normalized to unity collides elastically off an array of obstacles, whose centres lie on a square lattice. The scatterers have radius, R, whereas the lattice constant is unity.
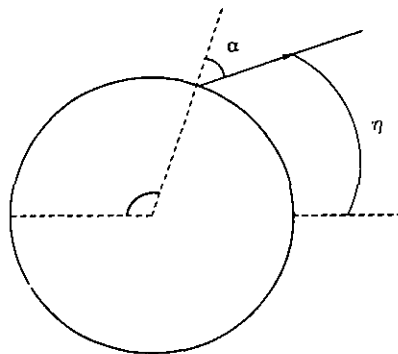

Figure 2. Description of coordinates (α, η) which represent the particle immediately after the collision with a scatterer. The angle η defined in the text is the angle the outgoing ray makes with the x-axis.

rational approximant to $\tan(\eta)$, where the accuracy is determined by the order R. Thus we must carry out a systematic search of the rationals in order to find the appropriate M, N satisfying (2).

Due to the symmetries of the lattice, we may always assume without loss of generality $0 \leq \tan \eta \leq 1$. Clearly then the collision criterion (3) implies that the integer pair (M, N) which we seek satisfy $M \geq N \geq 0$. Let us construct a 'tree' of all pairs of relatively prime integers as in figure 3. Each element is found by summing the coordinates of the left and right parent in the tree. This is exactly the same scheme as the popular Farey tree, where, in that scenario, the relatively prime integers are written as a fraction. Starting from the top, we check row by row whether a scatterer designated by an element of the row in the tree satisfies the condition of (3). Suppose that the $(2, 1)$ scatterer does not satisfy (3), for example. The sign of S yields the choice of scatterer in the next row which might satisfy the collision criterion. This algorithm of left and right moves and checks through the tree always gives the correct scatterer for SSB.
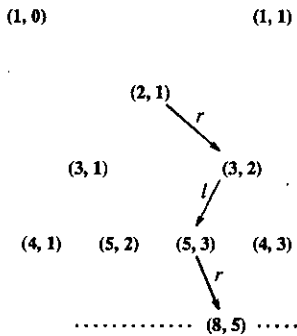

Figure 3. Motion down the tree which is used to find which is the next scatterer in the billiard simulation. Each (relatively prime) integer pair represents a scatterer and the pair is formed by adding the two parents of the tree as described in the text.

Suppose that as we proceed down the tree we count how many times in the same direction (successive left or right passes through the tree) we move before we switch directions. This number is precisely one of the digits of the continued fraction expansion for $\tan \eta$. Our rigorous assertions are then as follows. (i) The closest scatterer, (M, N), satisfying (3) has N/M as convergent or intermediate convergent of the continued fraction expansion of $\tan \eta$. (ii) If a convergent or intermediate convergent of $\tan \eta$ satisfies (3) then so do all larger intermediate convergents and convergents.

Our computer algorithm expands $\tan \eta$ in its continued fraction expansion, then checks convergents until a convergent satisfies (3). Then the program backtracks through the intermediate convergents to find the intermediate convergent whose preceding intermediate convergent no longer satisfies (3). At that point the computer has found the next scatterer.

We now argue that this representation of the motion is an encoding of the trajectory which is near optimal. Suppose we wish to carry out I (an integer such as 400 000) iterations of the billiard system, and we wish to ask how the optimal encoding varies with the scatterer radius, R. Towards this end, we may use the Brudno result. For the billiard system, the entropy of the map relating the angles of successive collisions has been extensively analysed and is conjectured to be asymptotically (for small R)

$$h_{\text{billiard}}(R) = -2 \log R + \text{constant} + \mathcal{O}(R). \tag{4}$$

Equation (4) has been well verified numerically and is analytically partly explained [3] (actually we present here the most accurate numerical verification of (4)). The Brudno theorem, equation (1), states that the shortest encoding length (in binary) should be the integer I times the entropy plus a constant independent of I. We can explicitly give an encoding which has a length I times $(-2 \ln R + \text{constant} = h(R) + \text{constant})$. Since our encoding has the same singular behaviour (for small R) as the entropy and the optimal encoding, then we would say that our algorithm is 'near optimal'.

One can arrive at the near-optimal estimate of the preceding paragraph by a simple argument. Let us assume that the correlations between collisions in a billiard simulation are negligible (which is a very reasonable assumption† particularly for small R). Then we should expect that the entropy should be the smallest amount of information necessary to find the next scatterer in the same procedure that we have described above. If there are on average $\langle k(R) \rangle$ digits of the continued fraction expansion for $\tan \eta$ necessary to find the next scatterer, then the amount of information contained in those $\langle k(R) \rangle$ digits should be $\langle k(R) \rangle$ times the entropy of the continued fraction algorithm, $h_{\text{cont fraction}}$. This latter constant is the average amount of information contained in each digit. Thus:

$$h_{\text{billiard}}(R) = h_0 + h_{\text{cont fraction}}\langle k(R) \rangle + \mathcal{O}(R). \tag{5}$$

Here h_0 is a constant independent of R. The quantity $h_{\text{cont fraction}}$ may be calculated from the continued fraction shift map, $T(x) = 1/x - [1/x]$, where $[\omega]$ represents the largest integer smaller than ω. This map supports the invariant (Gauss) measure on $[0, 1]$:

$$d\mu(x) = \frac{1}{\log 2} \frac{1}{x+1}.$$

The entropy for this one-dimensional map is well known:

$$h_{\text{cont fraction}} = \int d\mu(x) \log \left| \frac{dT(x)}{dx} \right| = \frac{\pi^2}{6 \log 2} \approx 2.373\ 138.$$

Let us recap. We claim that there is an extremely efficient method for simulating the dynamics of the point particle moving through a lattice of scatterers by expanding one of the collision angles in its continued fraction expansion. Moreover when we store the history of the dynamics by keeping track of these same digits we are able to achieve a near optimal encoding of the trajectory. That is the complexity-entropy as would be calculated from this method differs from that of the Brudno complexity-entropy (the lower bound) by a mere constant independent of R, which is irrelevant in comparison with the singular term $-2 \ln R$ which dominates the behaviour of both quantities. It is noteworthy to point out that the continued fraction expansion, for approximating a real number, is the best approximation scheme (in many senses [5]) and has positive entropy. Thus it is entirely reasonable that the continued fraction encoding should be near optimal.

Our numerical results are shown in table 1. The averages were found by following a trajectory through 400 000 collisions (with double precision arithmetic). The radius

† For any R the system is Bernoulli (see [4]).

Table 1. The results for a billiard simulation. A trajectory is followed through 400 000 collisions using double precision arithmetic. The average number of digits of the continued fraction expansion for tan η needed to calculate the successive scatterer, the average sum of these digits, and the various entropies are all tabulated.

$\log_{10} R$	Average number of digits	Average sum of digits	$h_{\text{billiard}} + 2 \log R$	$(h_{\text{billiard}}/h_{\text{cont frac}}) - \langle k \rangle$	CPU time (s)
−1.0	1.343	3.04	−0.0393	0.581	201.6
−2.0	3.189	13.44	−0.1171	0.642	277.7
−3.0	5.122	30.35	−0.1284	0.646	357.6
−4.0	7.054	53.86	−0.1314	0.653	437.0
−5.0	8.998	85.54	−0.1276	0.651	517.3
−6.0	10.941	115.89	−0.1265	0.646	598.0

of the scatterers was varied through six decades (and can be lowered even further with only mild technical difficulties). If we let

$$x = \log_{10} \frac{0.1}{R} \tag{6}$$

we find the following approximate relations. The average number of digits of the continued fraction expansion of the tangent of the angle η needed is approximately

$$\langle k(R) \rangle = 1.24 + 1.194x. \tag{7}$$

The average sum of these digits per collision are

$$\text{average sum of digits} \simeq 3.0 + 7.15x + 3.25x^2. \tag{8}$$

The sum of the digits corresponds to the level of the tree which is reached before a collision takes place (figure 3). At this point, we should state how the entropy $h(R)$ is computed. One may write down an expression for J_+, the Jacobian of the billiard map (which maps collision angles to collision angles) as in (2) (see for example [3, 6]), and equate time and phase averages via the Birkhoff ergodic theorem to arrive at the numerical result for the entropy. We see from the table that the entropy of the billiard system and the entropy of the continued fraction expansion (for both of which we have analytical results) satisfy the following relation with the average number of digits

$$\langle k(R) \rangle = \frac{h_{\text{billiard}}(R)}{h_{\text{cont fraction}}} - 0.648. \tag{9}$$

Equation (9) is our main numerical result and substantiates the heuristic analytical argument of (5). We should stress that the left- and right-hand sides of (9) are independently determined. The analytical asymptotic form for the entropy, $h(R)$, of $\text{SSB}(R)$ has been discussed first in [3], whereas $h_{\text{cont fraction}}$ is simply a constant whose value has been given above. On the other hand the quantity $\langle k(R) \rangle$ has been evaluated for the first time in this work. Thus as $R \to 0$, the average number of digits from tan η necessary to find successive scatterers holds a linear relationship with the billiard entropy divided by a known constant: the information contained in the digits of tan η is equal to the information contained in the dynamics of the map, modulo a constant independent of R.

There are practical implications to our method and result. The continued-fraction method enables us to study the Sinai system for much smaller radii than has been done previously [3, 6, 7]. Conventional algorithms have the simulation time scale with the mean free time of the point particle between collisions:

$$\langle \tau(R) \rangle = \frac{1}{2R} - \frac{\pi R}{2}. \tag{10}$$

Using the same definition of x as in (9) we see that our CPU time depends in the following way on the scatterer radius

$$\text{CPU time} = (198 + 80x)\ \text{s}. \tag{11}$$

Thus our fourfold increase in CPU time from table 1 for decreasing the scatterer radius by five orders of magnitude should be compared with the expected 10^5-fold (i.e. intractable) increase in CPU time one needs in a conventional simulation. Our method also immediately extends to other lattice type configurations (for example, we may map a twelfth sector of a triangular lattice to the eighth sector of a rectangular lattice, and thus apply our method). We are indeed optimistic that the continued fraction method should provide a useful analytical tool to study various properties† of the billiard system, which has been used as a caricature of non-equilibrium systems [8].

The relation (1) also may be interpreted from a practical point of view. Consider a straightforward algorithm for carrying out a billiard simulation such that the amount of time needed to simulate a fixed number of collisions scales with the mean free time, and hence as $(1/2R)$ for R small as $R \to 0$. Since we know the entropy behaves as $-2 \ln R$ in this limit, then we note that the ratio of the simulation time to the entropy becomes arbitrarily large as R is lowered. Since the system is Bernoulli for any R (that is, there are no special Feigenbaum points or other subtleties), we should expect that the simulation time should be related to the optimal encoding length [9]‡. That is, if we have a good way to encode the system dynamics, it is very reasonable that we should be able to use the same calculations that transcribes the dynamics into the encoding to calculate the particle trajectory (as we have done here). Thus we should suspect that if the simulation time and the entropy do not behave analytically in the same fashion as a control parameter is varied, that there *must* be a better simulational technique.

There are two notions of entropy discussed in this paper. The metric entropy (measure of randomness) may be measured by an observer outside of the system who measures the rate of expansion of the fibres in the phase space. The algorithmic entropy measures the length of the optimal encoding of the system dynamics. In general only the former is a readily measurable quantity, and not the latter. This is because there is in general no procedure for constructing the optimal encoding. Brudno showed theoretically that the algorithmic entropy is never larger than the metric entropy, and that in general they are equal. We have demonstrated the equality of Brudno by explicitly constructing the encoding. In order to do so, we needed to identify the most important feature of the dynamical system: there is a slope $(\tan \eta)$ which is being approximated by rational convergents. The relation (9) shows us that the encoding we have explicated above is nearly optimal.

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† For example, the lack of the existence of a diffusion constant for SSB can be shown to be related to the number theoretic properties of the lattice and may therefore be analysed via a continued fraction approach.
‡ Some difficulties in identifying different types of complexity are discussed in [9].

It is interesting that one can arrive at certain of the above results by merely examining the linear (trivial) part of the motion. It is known that the discontinuities of the system give a very natural way to form a partition for a dynamical system. In this system, the dynamics (for small R) are essentially completely uncorrelated from collision to collision, so that the dynamical entropy turns out to be essentially the information theoretic entropy of the partition. However, the author feels that the mechanism by which discontinuities give rise to random behaviour has still not been completely explained. As a further rather speculative remark, it seems likely to the author that, for an arbitrary dynamical system, there exists no algorithm for constructing a Markov partition.

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